DCJ-003-1164002 Seat No. ________

M. Sc. (Sem. IV) (CBCS) Examination

July – 2022

Mathematics : CMT-4002

(Integration Theory)

Faculty Code : 003
Subject Code : 1164002

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions :

(1) Each question carries 14 marks.

(2) There are 5 questions in total.

1 Answer the following questions : **14**

(a) Define : Counting Measure. Also find the counting measure of a set $A = [0,2022] \cap \mathbb{N}$.

(b) Give the statement of Radon – Nikodym Theorem for signed measure.

(c) Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \gamma)$ be complete measure spaces. Then prove that, $R = \{A \times B \subset X \times Y / A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ is a semi-algebra.

(d) Prove that : A function is continuous if and only if it is lower semi continuous as well as upper semi continuous.

(e) If μ^* is an outer measure on a set X and $\beta \in P(X)$ be such that $\mu^* \beta = 0$ then prove that, β is μ^* measurable.

(f) Prove that, every compact subset of K of a Hausdorff space is closed.

(g) Define terms : σ-compact, Bounded, σ-Bounded sets.

2 Answer any **two** questions : **14**

(a) Define : σ-algebra of subset of a set *X*. If *X* is any set $x_0 \in X$ then prove that, $\mu : P(X) \to \{0,1\}$ defined by

$$\mu(A) = \begin{cases} 1 & ; \text{if } x_0 \in A \\ 0 & ; \text{if } x_0 \in X - A \end{cases}$$

is measure on $(X, P(X))$.

(b) Define : Measure absolutely continuous with respect to another measure and mutually singular measures. If (X, A) is a measurable space and γ, μ are signed measures on $(X, \mathrm{A}), \gamma \perp \mu, \gamma \ll \mu$ then prove that $\gamma = 0$.

(c) Let γ be a signed measure on (X, A). Prove that, $\exists$ unique measures γ^+ and γ^- on (X, A) such that $\gamma = \gamma^+ - \gamma^-$ on $\mathrm{A}, \gamma^+ \perp \gamma^-$, where γ^+ and γ^- are positive and negative part of γ respectively.

3 Answer the following questions : **14**

(a) Let μ be a measure on algebra A of subset of a set *X* and μ^* be the outer measure on X induced by μ. Prove that, every element of A is μ^* – measure.

(b) If μ^* is an outer measure on a set *X* and $B = \{E \subseteq X / E$ is μ^* – measurable$\}$. Prove that, *B* is σ-algebra of subset of *X*.

OR

3 Answer the following questions : **14**

(a) Let (X, A, μ) be a finite complete measure space, *p*, *q* be extended nonnegative real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, *g* be integrable on (X, A, μ) and $\left| \int_X g \phi \, d\mu \right| \leq M . \| \phi \|_p$, for all simple measurable function ϕ on *X* for some $M > 0$. Prove that, $g \in L^q(\mu)$.

(b) If X is a countable set and μ is the counting measure on $(X, P(X))$. Prove that, $L^p(\mu) \cong l^p, 1 \le p \le \infty$.

4 Answer any **two** questions : **14**

(a) State and prove : Carthrodry Extension Theorem.

(b) Let (X, A) be a measure space and $f : X \to [0, \infty]$ be a measurable. Prove that, $\exists$ a sequence $\{s_n\}_{n=1}^{\infty}$ of non – negative simple function on (X, A) such that

1. $s_1 \le s_2 \le \ldots \le s_n \le \ldots \le f$ on X.
2. $\lim_{n \to \infty} s_n(x) = f(x), \forall x \in X$.

(c) Let $\mathbb{C}$ be a semi algebra of subset of a set S and $\mu : \mathbb{C} \to [0, \infty]$ be such that

1. $c \in \mathbb{C}, c = \bigcup_{i=1}^{n} c_i$. Prove that,
   $\mu(c) = \sum_{i=1}^{n} \mu(c_i), \forall n \in \mathbb{N}, c_i \in \mathbb{C}$ and
   $c_i \bigcap c_j = \phi, \forall i, j$.
2. $c \in \mathbb{C}, c = \bigcup_{n=1}^{\infty} c_n$. Prove that,
   $\mu(c) = \sum_{n=1}^{\infty} \mu(c_n), \forall c_i \in \mathbb{C}$ and $c_i \bigcap c_j = \phi, \forall i, j$.

5 Answer any **two** questions : **14**

(a) Let X be a locally compact T_2 – space, Let K be a compact G_δ – set in X. Prove that, $\exists f \in C_c(X)$ such that

1. $f(x) = 1, \forall x \in K$
2. $0 \le f(x) < 1, \forall x \in X - K$

(b) Define : Baire measure on the real line. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and continuous function on the right. Prove that, $\exists$ a baire measure μ on the real line such that $\mu(a, b] = f(a) - f(b), \forall a, b \in \mathbb{R}$ and $a < b$.

(c) Let X be a locally compact separable metric space. Prove that, $B_0(X) = B_a(X)$.

(d) Prove that : Let X be a topological space.

1. For $U \subseteq X$, $\chi_U : X \to \{0,1\}$ is lower semi continuous if and only if U is open in X.

2. If $f_\alpha : X \to \{0,1\}$ are lower semi continuous, $\forall \alpha \in \Lambda$ then prove that $\sup_{\alpha \in \Lambda} f_\alpha$ is also lower semi continuous on X.
